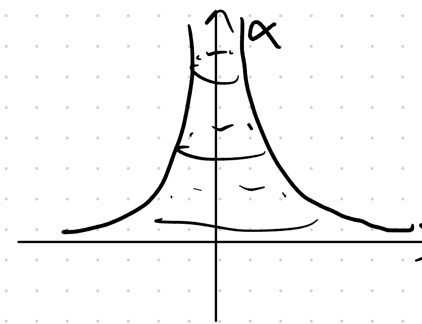
2/11/23

MATH4030 Tutorial

Announcements:

- Midterm: Mean: 64, Median: 66, SD: 19
- HW3 Solution will be posted tomorrow morning, HW2 aim to finish grading this week.
- HW4, due 13/11.



Ex: Pseudosphere, an example of a surface S with $K \equiv -1$

Consider the tractrix: $\alpha : (0, \frac{\pi}{2}) \to xz$-plane by

$$\alpha(t) = (\sin t, \cos t + \log\tan\tfrac{t}{2}).$$

Take the surface of revolution obtained by rotating α about the z-axis.

So S is parameterized by

$$X(t,\theta) = (\sin t\cos\theta, \sin t \sin\theta, \cos t + \log\tan\tfrac{t}{2}).$$

By computing E, F, G, e, f, g and using $K = \frac{eg-f^2}{EG-F^2}$, you can show that $K \equiv -1$.

Def: a surface S is called minimal if $H \equiv 0$. (critical point of area functional).

- a parametrization $X(u,v)$ is isothermal if

$$|X_u| = |X_v| = \lambda, \quad \langle X_u, X_v \rangle = 0$$

$$\lambda \in \mathbb{R}.$$

(to be proved in lecture).

Thm: S is minimal iff $\exists$ isothermal param. X s.t. $\underbrace{X_{uu} + X_{vv} = 0}_{\Delta X = 0}$.

$\Delta X = 0 \leftarrow$ PDE (Complex Analysis condition

- $f, g : \mathbb{R}^2 \to \mathbb{R}$ satisfy CR-equations if

$$\frac{\partial f}{\partial u} = \frac{\partial g}{\partial v}, \quad \frac{\partial f}{\partial u} = -\frac{\partial g}{\partial v}.$$

From Complex Analysis, we know $h: f + ig$ is a holomorphic function.

f, g are called harmonic conjugates.

- Let X, Y be isothermic param. of minimal surfaces M, N s.t. the component function are pairwise harmonic conjugate, then M, N are called conjugate

<u>minimal surfaces</u>.

<u>Q1</u>: Show that the catenoid, helicoid are conjugate minimal surfaces.

Catenoid: $X(u,v) = (\overset{f_1}{\cosh v \cos u}, \overset{f_2}{\cosh v \sin u}, \overset{f_3}{v})$

Helicoid $Y(u,v) = (\overset{g_1}{\sinh v \sin u}, \overset{g_2}{-\sinh v \cos u}, \overset{g_3}{u})$

Basically, check:
$X_v = Y_u$,
$X_u = -Y_v$.

Pf: Easy to check that $X_{uu} + X_{vv} = 0$, $Y_{uu} + Y_{vv} = 0$ and that X, Y are isothermal param.

$X_u = (-\cosh v \sin u, \cosh v \cos u, 0)$

$Y_v = (\cosh v \sin u, -\cosh v \cos u, 0)$

$\Rightarrow X_u = -Y_v$. So they indeed are conjugate param.

$X_v = (\sinh v \cos u, \sinh v \sin u, 1)$

$Y_u = (\sinh v \cos u, \sinh v \sin u, 1)$.

$\Rightarrow X_v = Y_u$.

Q2: Given two conjugate minimal surfaces X, Y, show that the surface

$$Z_t = \cos t\, X + \sin t\, Y$$

is minimal for all $t \in \mathbb{R}$.

Pf: We'll show Z_t is isothermal and $\Delta Z_t = 0$.

$\Delta Z_t = 0$: $Z_{uu} = \cos t\, X_{uu} + \sin t\, Y_{uu}$

$Z_{vv} = \cos t\, X_{vv} + \sin t\, Y_{vv}$

Since X, Y satisfy $\Delta X = 0$, $\Delta Y = 0$, we have

$$\Delta Z = Z_{uu} + Z_{vv} = \cos t\,(\cancel{X_{uu} + X_{vv}}^{\,0}) + \sin t\,(\cancel{Y_{uu} + Y_{vv}}^{\,0}) = 0.$$

Remains to check Z_t is an isothermal param.

$Z_u = \cos t\, X_u + \sin t\, Y_u$, $Z_v = \cos t\, X_v + \sin t\, Y_v$.

$$\langle Z_u, Z_v \rangle = \cos^2 t \,\cancel{\langle X_u, X_v\rangle}^{\,0} + \sin t \cos t \,\langle X_u, Y_v\rangle + \sin t \cos t\, \langle Y_u, X_v \rangle$$
$$+ \sin^2 t\, \cancel{\langle Y_u, Y_v \rangle}^{\,0}$$ since X, Y are isothermal

$$= \sin t \cos t \langle X_u, Y_v\rangle + \sin t \cos t \langle Y_u, X_v\rangle .$$

CR equis $\Rightarrow$ $= \sin t \cos t \langle X_u, X_u\rangle - \sin t \cos t \langle X_v, X_v\rangle$

$$= \sin t \cos t \left(\cancel{|X_u|^2 - |X_v|^2} \right) = 0$$

$\uparrow$

X is isothermal

$$|Z_u|^2 = \langle Z_u, Z_u\rangle = \langle \cos t X_u + \sin t Y_u , \cos t X_u + \sin t Y_u\rangle$$

$$= \cos^2 t |X_u|^2 + 2 \sin t \cos t \langle X_u, Y_u\rangle + \sin^2 t |Y_u|^2$$

$$= \cos^2 t |X_u|^2 + 2 \sin t \cos t \cancelto{0}{\langle X_u, -X_v\rangle} + \sin^2 t |\underset{\substack{\| \\ -X_v}}{Y_u}|^2$$

$$= \cos^2 t |X_u|^2 + \sin^2 t |X_v|^2$$

$$= \lambda^2 . \quad \text{where } \lambda = |X_u| = |X_v| .$$

Similar argument to show $|Z_v|^2 = \lambda^2$.

So Z_t is isothermal and by the thm Z_t is minimal for all t

So we have obtained a one-param. family of minimal surfaces

$$Z_t = \cos t \begin{pmatrix} \cosh v \cos u \\ \cosh v \sin u \\ v \end{pmatrix} + \sin t \begin{pmatrix} \sinh v \sin u \\ -\sinh v \cos u \\ u \end{pmatrix}$$

Z_t is minimal for all t, Z_0 is the catenoid, $Z_{\frac{\pi}{2}}$ is the helicoid.